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Non-linear critical relaxation in the kinetic Ising model

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Abstract. We show, by calculating various linear and non-linear relaxation times exactly, that the scaling law $\Delta_\psi^{(nl)} = \Delta_\psi^{(l)} - \beta_\psi$ is valid for the one-dimensional kinetic Ising model. In the two-dimensional case, the critical exponent of the non-linear relaxation time of the order parameter is estimated from an eleventh-order high-temperature series expansion. The result, $\Delta_M^{(nl)} = 2{\cdot}00 \pm 0{\cdot}04$, is in agreement with the scaling prediction.

1. Introduction

The dynamics of second-order phase transitions poses many interesting problems from both the experimental and the theoretical points of view (Hohenberg and Halperin 1977). A remarkable aspect of critical-slowing-down experiments is the non-linear relaxation: near the critical point the time evolution becomes so slow that in some materials one can examine the details of the relaxation far from equilibrium (Collins and Teh 1973, Sato and Hirakawa 1977). These experiments provide us with new information about non-equilibrium processes, so we can gain insight in a field which is still in the initial stages of development (Binder 1976).

In purely relaxational systems an overall characterisation of the relaxation far from equilibrium may be given by the non-linear relaxation time (Suzuki 1971):

$$\tau_\psi^{(nl)} = \int_0^\infty \psi(t)/\psi(0)\,\mathrm{d}t, \tag{1}$$

where ψ is a non-conserved quantity with zero equilibrium average. The first question one would like to answer is how $\tau_\psi^{(nl)}$ relates to the corresponding linear relaxation time, i.e. to the decay time of the equilibrium fluctuations

$$\tau_\psi^{(l)} = \int_0^\infty \langle\psi(t)\psi(0)\rangle/\langle\psi^2\rangle\,\mathrm{d}t. \tag{2}$$

It has been shown that the critical point singularities of $\tau_\psi^{(nl)}$ and $\tau_\psi^{(l)}$ are different (Rácz 1975) and the corresponding critical exponents $\Delta_\psi^{(nl)}$ and $\Delta_\psi^{(l)}$ are related by (Rácz 1976, Fisher and Rácz 1976)

$$\Delta_\psi^{(nl)} = \Delta_\psi^{(l)} - \beta_\psi, \tag{3}$$

where β_ψ is the exponent characterising the scaling of ψ with respect to temperature (e.g. $\beta_\psi = \beta$ if $\psi = M$ is the order parameter and $\beta_\psi = 1 - \alpha$ if $\psi = E$ is the energy).

The scaling law (3) is based on the assumptions that dynamic scaling can be extended to non-linear phenomena and that for a quantity ψ the extent of the linear

regime of relaxation is scaled with the static exponent β_ψ. Renormalisation group calculations (Bausch and Janssen 1976, Suzuki 1976) have proved the first assumption to be correct for the order parameter relaxation (restricted scaling in the terminology of Halperin and Hohenberg 1969). The validity of the second assumption, however, remains an open question, so by examining (3) one actually investigates how the extent of the linear regime of relaxation is scaled, and also whether extended scaling applies to non-linear relaxation.

Up to now the scaling law (3) has been proved for the dynamical droplet model (Kretschmer *et al* 1976) and for the spherical limit of the time-dependent Ginzburg–Landau model (Rácz and Tél 1977). There are also Monte Carlo calculations (Stoll *et al* 1973, Kretschmer *et al* 1976) and high-temperature series expansions (Rácz and Collins 1976, Ikeda 1976a, b, White 1976 and unpublished) for the non-linear energy and order parameter relaxation times $\tau_E^{(\mathrm{nl})}$ and $\tau_M^{(\mathrm{nl})}$ of the kinetic Ising model.

Our work was motivated by the problems in the latter studies. In the case of the two-dimensional kinetic Ising model, both the Monte Carlo work and the high-temperature series yield a value for the exponent $\Delta_E^{(\mathrm{nl})}$ of $\tau_E^{(\mathrm{nl})}$ which is smaller than the scaling prediction. Furthermore, the high-temperature series are not long enough to demonstrate convincingly the small difference between the critical exponents of $\tau_M^{(\mathrm{l})}$ and $\tau_M^{(\mathrm{nl})}$ ($\Delta_M^{(\mathrm{l})} - \Delta_M^{(\mathrm{nl})} = \beta = \frac{1}{8}$).

In order to improve our knowledge about the dynamics of the kinetic Ising model, first we examined the energy and order parameter relaxations in the one-dimensional case. The one-dimensional Ising model exhibits only a zero-temperature ordering. Critical exponents for this transition can nevertheless be defined, since quantities like susceptibility, correlation length, etc display power law singularities if they are expressed as functions of the natural high-temperature variable

$$v = \tanh(J/k_B T), \tag{4}$$

where J is the spin–spin interaction strength and T is the temperature. In § 2 we show that the zero-temperature singularity of the dynamical quantities can also be characterised by critical exponents and the scaling law (3) holds for both the order parameter and energy relaxations. The analytical structure of the different relaxation times, relevant to the analysis of high-temperature series, is also discussed briefly in § 2.

In the more interesting two-dimensional case, we have extended up to eleventh order the high-temperature series of $\tau_M^{(\mathrm{nl})}$ (§ 3). Analysing the series by Padé approximant methods, the critical exponent of $\tau_M^{(\mathrm{nl})}$ is found to be $\Delta_M^{(\mathrm{nl})} = 2{\cdot}00 \pm 0{\cdot}04$, in agreement with the scaling prediction.

2. Relaxation in the one-dimensional kinetic Ising model

The one-spin-flip kinetic Ising model (Glauber 1963) is one of the simplest systems where non-trivial critical-slowing-down effects can be investigated. The static properties of the model are well known and the dynamics is also simple, in the sense that it is purely relaxational and there are no conserved quantities. Thus the only complication one has to cope with is the critical point divergence of the relaxation times.

Since the model has been discussed extensively in the literature, it will not be described here. An interested reader may consult Glauber's original paper (Glauber 1963), where the one-dimensional case is treated in detail. In this section we use

Glauber's exact results to establish the validity of (3) for the one-dimensional kinetic Ising model.

The order parameter relaxation is trivial, since in one dimension the magnetisation (order parameter) decays exponentially. This means that both the linear and non-linear relaxation times are equal to the decay time calculated by Glauber (1963):

$$\tau_M^{(l)} = \tau_M^{(nl)} = (1+v^2)/(1-v)^2 \tag{5}$$

where v is given by (4) and the constant setting the time scale is chosen to be unity.

The critical point is at $v_c = 1$, so one can see from (5) that $\Delta_M^{(l)} = \Delta_M^{(nl)} = 2$. This is in accord with the scaling prediction, since in one dimension $\beta = 0$ (Fisher 1974).

The energy relaxation is a bit more complicated. The time evolution of the energy towards its equilibrium value v is described by the following sum (Glauber 1963):

$$E(t) - v = e^{-2t} \sum_{k=1}^{\infty} (v_0^k - v^k) Q_k(2\gamma t), \tag{6}$$

where the energy is measured in units of J, v_0 is the initial value of E, $\gamma = 2v/(1+v^2)$ and

$$Q_k(x) = I_{k-1}(x) - I_{k+1}(x), \tag{7}$$

$I_k(x)$ being the imaginary-argument Bessel function (Watson 1958).

Now, the equilibrium relaxation function $\phi_E^{(l)}$ is found from (6) by dividing by the initial deviation $v_0 - v$ and taking the limit of zero initial deviation:

$$\phi_E^{(l)}(t) = \lim_{v_0 \to v} \frac{E(t)-v}{v_0 - v} = e^{-2t} \sum_{k=1}^{\infty} k v^{k-1} Q_k(2\gamma t). \tag{8}$$

The non-linear relaxation can be investigated by choosing, e.g., a completely random ($v_0 = 0$) far from equilibrium initial state. Then the non-linear relaxation function is expressed as

$$\phi_E^{(nl)}(t) = \left.\frac{E(t)-v}{v_0 - v}\right|_{v_0=0} = e^{-2t} \sum_{k=1}^{\infty} v^{k-1} Q_k(2\gamma t). \tag{9}$$

Note that the usual, completely ordered ($v_0 = 1$) initial state cannot be used now, because then the initial deviation $1 - v$ would go to zero when approaching the critical point.

The area under the relaxation curve gives the relaxation time, so the calculation of $\tau_E^{(l)}$ and $\tau_E^{(nl)}$ requires the evaluation of the following integral:

$$\int_0^{\infty} e^{-2t} Q_k(2\gamma t)\, dt = v^{k-1}(1+v^2)/2. \tag{10}$$

Using (10), the integrals of (8) and (9) result in simple series which are summed to yield

$$\tau_E^{(l)} = \frac{1+v^2}{2(1-v^2)^2} \sim (1-v)^{-2} \tag{11}$$

and

$$\tau_E^{(nl)} = \frac{1+v^2}{2(1-v)} \sim (1-v)^{-1}. \tag{12}$$

The result for $\tau_E^{(l)}$ has already been calculated by Felderhof and Suzuki (1971).

From (11) and (12) we obtain the critical exponents

$$\Delta_E^{(l)} = 2 \qquad \text{and} \qquad \Delta_E^{(nl)} = 1 \tag{13}$$

which agree again with the scaling prediction since in one dimension the deviation of the energy from its critical point value scales like $1-v$, i.e. $\beta_E = 1$.

In connection with Monte Carlo calculations and high-temperature series, the first time moment of the relaxation function has been introduced as a new relaxation time (Suzuki 1970, Schneider and Stoll 1974)

$$\tau_{\psi 1} = \left(\int_0^\infty t \phi_\psi(t)\, \mathrm{d}t \right)^{1/2}. \tag{14}$$

The linear and non-linear relaxations can again be distinguished and now the same arguments which lead to (3) result in a different scaling law (Ikeda 1977):

$$\Delta_{\psi 1}^{(nl)} = \Delta_{\psi 1}^{(l)} - \beta_\psi / 2. \tag{15}$$

This scaling law can also be verified for the one-dimensional kinetic Ising model. The case of order parameter relaxation does not require any calculation, since the equalities $\tau_{M1}^{(nl)} = \tau_{M1}^{(l)} = \tau_M^{(l)}$ follow from the exponential relaxation of the magnetisation. The calculation of $\tau_{E1}^{(l)}$ and $\tau_{E1}^{(nl)}$ is carried out similarly to that of $\tau_E^{(l)}$ and $\tau_E^{(nl)}$, and we obtain

$$\tau_{E1}^{(l)} = \frac{(1+v^2)^{3/2}}{2(1-v^2)^2} \sim (1-v)^{-2} \tag{16}$$

and

$$\tau_{E1}^{(nl)} = \frac{1+v^2}{2(1-v^2)^{3/2}} \sim (1-v)^{-3/2}. \tag{17}$$

Since $\beta_E = 1$, the exponents $\Delta_{E1}^{(l)} = 2$ and $\Delta_{E1}^{(nl)} = \frac{3}{2}$ satisfy (15).

The first time moment of the relaxation function has been investigated because of its apparent smooth high-temperature series (Ikeda 1977). Expanding (11), (12), (16) and (17), we find that the ratio estimates of the critical exponents, $\Delta_{E1}^{(l)}$ and $\Delta_{E1}^{(nl)}$, are indeed smooth sequences (table 1), but in a given order the deviation from the exact value is always larger than that of the corresponding Δ_E. So, the results obtained from

Table 1. Ratio estimates of the critical exponents of the various energy relaxation times in the one-dimensional kinetic Ising model, calculated from the high-temperature expansions of (11), (12), (16) and (17). Note that the expansion variable is v^2, so this table requires knowledge of the high-temperature series up to v^{12}.

	$\Delta_E^{(l)}$	$\Delta_{E1}^{(l)}$	$\Delta_E^{(nl)}$	$\Delta_{E1}^{(nl)}$
1	3·00	3·50	2	2·50
2	2·33	2·64	1	1·70
3	2·20	2·34	1	1·61
4	2·14	2·31	1	1·58
5	2·11	2·23	1	1·56
6	2·09	2·19	1	1·55
Exact	2	2	1	1·5

the high-temperature series of the first time moment have to be interpreted with caution, especially if the series are short.

Having obtained the different relaxation times exactly, it is worth discussing their analytical structure. It might give a clue about higher dimensions, i.e. it might help to analyse the high-temperature series in two and three dimensions.

The energy relaxation does not depend on the sign of the interaction between the neighbouring spins. Accordingly, the energy relaxation times are singular at both the ferromagnetic ($v_c = 1$) and antiferromagnetic ($v_c' = -1$) critical points. On the other hand, the magnetisation is not a relevant quantity in an antiferromagnet, so $\tau_M^{(1)}$ is singular only at $v_c = 1$.

It is important to note that $\tau_{E1}^{(1)}$ is non-analytic at $v = \pm i$, and all the relaxation times vanish at those points. This 'instantaneous relaxation' follows from the particular form of the spin-flip transition probability used in the kinetic Ising model (Glauber 1963):

$$w_i = \frac{1}{2}\left(1 - \frac{v}{1+v^2}\sigma_i(\sigma_{i+1} + \sigma_{i-1})\right), \tag{18}$$

where σ_i is the value of the ith spin. At $v = \pm i$ the transition probability is infinite and this leads to vanishing relaxation times. It may be expected in general that the singularities of the transition probability are reflected in the relaxation times.

The choice of transition probabilities is restricted only by the detailed balance condition. The above discussion shows, however, that one has to be cautious with the remaining arbitrariness. If the relaxation time is expanded around some point, then the singularity of the transition probabilities should not be nearer to that point than the physical singularity of the relaxation time.

Most frequently the expansion point is $v = 0$, i.e. the high-temperature series are considered. It may be observed from (5), (11), (12), (16), (17) and (18) that in one dimension the physical singularities and the singularities of the transition probabilities are on the circle of convergence of the high-temperature series. This leads to quite slow convergence of the ratio estimates of the critical exponents (table 1). A similar situation arises in two dimensions, as discussed in the next section.

3. Non-linear order parameter relaxation in two dimensions

The exact solution of the kinetic Ising model is available only in one dimension. An effective method of estimating its dynamic critical exponents in higher dimensions is the high-temperature series expansion (Suzuki *et al* 1969). High-temperature series have been derived for both the linear (Yahata and Suzuki 1969, Yahata 1971) and non-linear (Suzuki 1971, Rácz and Collins 1976, Ikeda 1976a, White 1976) relaxation times of the order in two and three dimensions. Short series have also been given for the energy relaxation times in two dimensions (Yahata 1971, Ikeda 1976b, White 1976 unpublished).

In this section the non-linear relaxation time $\tau_M^{(nl)}$ of the order is investigated in the square-lattice kinetic Ising model. This case is interesting because a relatively long (twelfth-order) series is available for the linear relaxation time $\tau_M^{(l)}$ (Yahata 1971) and its Padé analysis yields a very accurate value for the critical exponent $\Delta_M^{(l)} = 2{\cdot}125 \pm 0{\cdot}01$ (Rácz and Collins 1976). Thus $\Delta_M^{(nl)}$ remains the only unknown quantity in the scaling relation (3). Furthermore, it might be expected that, if the fluctuations

result in a deviation from (3), then the largest deviation can be observed in two dimensions because the kinetic slowing down is the largest in this case.

The critical exponent $\Delta_M^{(nl)}$ has been previously estimated from a ninth-order high-temperature series (Rácz and Collins 1976). The large uncertainty of the result ($\Delta_M^{(nl)} = 1{\cdot}95 \pm 0{\cdot}15$) and the small value of $\beta = \frac{1}{8}$, however, excluded the possibility of quantitative comparison with the scaling law (3). In order to determine $\Delta_M^{(nl)}$ more accurately, we have extended the high-temperature series of $\tau_M^{(nl)}$ up to eleventh order. Some details of the simple but quite voluminous computation may be found in the appendix. The final result is written as

$$\tau_M^{(nl)} = 1 + 4v + 16v^2 + \frac{148}{3}v^3 + \frac{416}{3}v^4 + \frac{10\,444}{27}v^5 + \frac{433\,264}{405}v^6 + \frac{3\,515\,524}{1\,215}v^7 + \frac{705\,704\,768}{91\,125}v^8 + \frac{1\,584\,670\,852}{76\,545}v^9 + 54\,887{\cdot}7211v^{10} + 143\,811{\cdot}2651v^{11} + \ldots . \tag{19}$$

The critical value of v is known ($v_c = \sqrt{2} - 1$), so applying the ratio method to (19), we find a sequence of unbiased estimates for the critical exponent of $\tau_M^{(nl)}$:

$$\Delta_M^{(nl)} = 1{\cdot}657, 2{\cdot}314, 1{\cdot}831, 1{\cdot}657, 1{\cdot}777, 1{\cdot}873, 1{\cdot}842, 1{\cdot}869, 1{\cdot}966, 1{\cdot}982, 1{\cdot}938 \ldots . \tag{20}$$

This sequence does not give $\Delta_M^{(nl)}$ accurately enough to say anything definite about the scaling prediction $\Delta_M^{(nl)} = 2{\cdot}0$. The reason for the slow convergence of the ratio estimates may be found in the transition probabilities:

$$w_i = \frac{1}{2}\left[1 - \sigma_i \tanh\left(\frac{J}{k_B T}\sum_a \sigma_{i+a}\right)\right], \tag{21}$$

where the sum is over the four nearest neighbours of σ_i. When all the neighbouring spins are in the same direction, the second term in (21) becomes proportional to $\tanh(4J/k_BT)$. Expressing $\tanh(4J/k_BT)$ in terms of $v = \tanh(J/k_BT)$, one can easily see that it is singular at $v = \pm i(\sqrt{2} - 1) = \pm iv_c$. Consequently, the relaxation time has additional singularities following from the particular form of the transition probabilities (see discussion in the previous section). These singularities are on the circle of convergence and they make the convergence of ratio estimates slow.

White (1976) introduced transition probabilities which satisfy the detailed balance condition and are free of the above problem. It is a promising way of improving the high-temperature series. Unfortunately, his transition probabilities make the computation more involved, thus one cannot go easily to higher orders in the expansion.

Since the ratio analysis does not yield the desirable accuracy, we turn to the more powerful Padé approximant methods (Hunter and Baker 1973). The most reliable value of $\Delta_M^{(nl)}$ is obtained by taking into account the correlation between the Padé estimates of the critical point and critical exponent (Meijer and Farrel 1975). The importance of these correlations may be seen from table 2, where the estimates of $\Delta_M^{(nl)}$ and v_c, resulting from the Padé approximants to $\mathrm{d} \ln \tau_M^{(nl)}/\mathrm{d}v$, are displayed. One can observe that the higher the critical point estimate, the higher the exponent estimate. Since almost all critical point estimates are lower than the exact value, this means that a conventional analysis of the Padé table underestimates $\Delta_M^{(nl)}$.

In order to take into account the correlations, we plot the exponent estimates against the critical point estimates (figure 1). Straight line extrapolation of the resulting smooth function gives at the exact critical point

$$\Delta_M^{(nl)} = 2{\cdot}00 \pm 0{\cdot}04, \tag{22}$$

Table 2. Padé-approximant estimates of the critical point $v_c = 0{\cdot}4142$ (upper numbers) and the critical exponent $\Delta_M^{(nl)}$ (lower numbers) of the non-linear relaxation time of the order in the square-lattice kinetic Ising model. Asterisks indicate the presence of poles which are in the immediate vicinity of $v = 0$ or at a distance less than $v_c/2$ from the physical singularity.

M \ L	2	3	4	5	6	7
3	0·3999	0·3966	0·3907	0·3948	0·4024	0·4097
	1·516	1·465	1·359	1·447	1.657	1·909
4	0·3970	0·4036	0·3931	0·3863	—	
	1·473	1·543*	1·407	1·300*		
5	0·3908	0·3940	0·3958	0·4098		
	1·365	1·425	1·462	1·861		
6	0·3997	0·4034	0·3835			
	1·570	1·670	1·357*			
7	0·4029	0·3989				
	1·670	1·550*				
8	0·4159					
	2·363*					

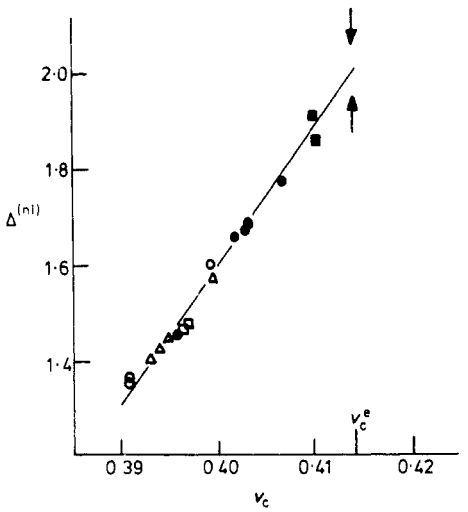

Figure 1. Correlation between the Padé estimates of the critical point v_c and the critical exponent $\Delta_M^{(nl)}$ calculated by retaining 10 (full squares), 9 (full circles), 8 (open triangles), 7 (open circles) and 6 (open squares) terms in the high-temperature series of $d \ln \tau_M^{(nl)}/dv$. We left out the estimates in which there is a pole in the immediate vicinity of $v = 0$ or at a distance less than $v_c/2$ from the physical singularity. The exact value of v_c is denoted by v_c^e.

which is in excellent agreement with the scaling prediction $\Delta_M^{(nl)} = 2{\cdot}0$. Of course, it would be desirable to reduce further the error bar in (22). The calculation of new terms in the high-temperature series, however, would take considerable effort and computer time, so probably it is more promising to approach this problem by renormalisation group methods.

4. Final remarks

The results of the previous two sections and the set of critical exponents for the three-dimensional kinetic Ising model ($\Delta_M^{(l)} = 1{\cdot}32 \pm 0{\cdot}03$ (Rácz and Collins 1976), $\Delta_M^{(nl)} = 1{\cdot}05 \pm 0{\cdot}05$ (Ikeda 1976a), $\beta = 0{\cdot}312 \pm 0{\cdot}001$ (Essam and Fisher 1963)), demonstrate convincingly that the scaling law (3) holds for the order parameter relaxation.

As to the energy relaxation, it is shown in § 2 that, in the one-dimensional kinetic Ising model, (3) is valid for the energy relaxation too. For the two-dimensional case the scaling prediction is $\Delta_E^{(nl)} = \Delta_E^{(l)} - 1 + \alpha = 1{\cdot}13 \pm 0{\cdot}01$. The high-temperature series for $\tau_E^{(nl)}$ (White 1976 unpublished, Ikeda 1976b) are so short (5–6 terms) that the somewhat small value of $\Delta_E^{(nl)}$, following from their ratio analysis, does not have to be taken seriously. At first sight the Monte Carlo calculation of Kretschmer *et al* (1976) also gives a number $\Delta_E^{(nl)} = 0{\cdot}9 \pm 0{\cdot}1$, which is in accord with the high-temperature series. One must remember, however, that the Monte Carlo studies (Stoll *et al* 1973) seem to underestimate all the dynamic exponents (e.g. $\Delta_E^{(l)} \approx 1{\cdot}85$ instead of $\Delta_E^{(l)} \approx 2{\cdot}13$). Thus the Monte Carlo calculations are in agreement with (3) if it is assumed that they underestimate both $\Delta_E^{(l)}$ and $\Delta_E^{(nl)}$ by the same amount.

So our final conclusion is that the scaling law (3) holds in every purely relaxational system which has been investigated so far. The existing small discrepancies can be attributed to the uncertainties in the values of the critical exponents.

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Appendix

Suzuki (1971) has described in detail how to perform high-temperature series expansion for $\tau_M^{(nl)}$ in the case of a square lattice. The work is essentially a graph counting exercise. It can be put on a computer and the series can be calculated up to an order limited only by the available computer time.

The results of our eleventh-order calculation are given below. Denoting the contributions of different classes of graphs by (for notation and details see Rácz and Collins 1976):

$$\hat{b}^k \hat{c}^l \ldots \hat{b}^n = (1-4a)\langle X_b^k X_c^l \ldots X_b^k S \rangle_0, \tag{23}$$

the contributions, which are new compared to the ninth-order results (Rácz and

Collins 1976), can be written as follows

$$\begin{aligned} \hat{b}^7 &= 39\,647{\cdot}575\,675\, a^7 c \\ \hat{b}^8 &= 163\,572{\cdot}723\,655\, a^8 c \end{aligned} \tag{24}$$

$$\begin{aligned} \hat{b}^4\hat{c}^1 &= 1\,447{\cdot}870\,348\, a^4 c^2 \\ \hat{b}^3\hat{c}^1\hat{b}^1 &= 1\,505{\cdot}207\,659\, a^4 c^2 \\ \hat{b}^2\hat{c}^1\hat{b}^2 &= 1\,283{\cdot}572\,763\, a^4 c^2 \\ \hat{b}^1\hat{c}^1\hat{b}^3 &= 1\,125{\cdot}132\,291\, a^4 c^2 \\ \hat{c}^1\hat{b}^4 &= 1\,068{\cdot}369\,822\, a^4 c^2 \end{aligned} \tag{25}$$

$$\begin{aligned} \hat{b}^5\hat{c}^1 &= 6\,332{\cdot}451\,703\, a^5 c^2 \\ \hat{b}^4\hat{c}^1\hat{b}^1 &= 6\,508{\cdot}644\,918\, a^5 c^2 \\ \hat{b}^3\hat{c}^1\hat{b}^2 &= 5\,691{\cdot}196\,722\, a^5 c^2 \\ \hat{b}^2\hat{c}^1\hat{b}^3 &= 5\,159{\cdot}125\,415\, a^5 c^2 \\ \hat{b}^1\hat{c}^1\hat{b}^4 &= 4\,448{\cdot}931\,180\, a^5 c^2 \\ \hat{c}^1\hat{b}^5 &= 4\,312{\cdot}201\,189\, a^5 c^2 \end{aligned} \tag{26}$$

$$\begin{aligned} \hat{b}^1\hat{c}^2 &= 58{\cdot}710\,494\, ac^3 \\ \hat{c}^1\hat{b}^1\hat{c}^1 &= 53{\cdot}651\,640\, ac^3 \\ \hat{c}^2\hat{b}^1 &= 50{\cdot}093\,827\, ac^3 \end{aligned} \tag{27}$$

$$\begin{aligned} \hat{b}^2\hat{c}^2 &= 262{\cdot}409\,255\, a^2 c^3 \\ \hat{b}^1\hat{c}^1\hat{b}^1\hat{c}^1 &= 240{\cdot}313\,577\, a^2 c^3 \\ \hat{c}^1\hat{b}^2\hat{c}^1 &= 221{\cdot}159\,979\, a^2 c^3 \\ \hat{b}^1\hat{c}^2\hat{b}^1 &= 219{\cdot}271\,617\, a^2 c^3 \\ \hat{c}^1\hat{b}^1\hat{c}^1\hat{b}^1 &= 215{\cdot}345\,514\, a^2 c^3 \\ \hat{c}^2\hat{b}^2 &= 192{\cdot}075\,250\, a^2 c^3 \end{aligned} \tag{28}$$

where a and c have the following high-temperature expansions:

$$a = v - 3v^3 + 15v^5 - 85v^7 + 493v^9 - 2871v^{11} \ldots \tag{29}$$

$$c = -2v^3 + 14v^5 - 84v^7 + 492v^9 - 2870v^{11} \ldots . \tag{30}$$

Substituting (29) and (30) in (24)–(28) and expanding the terms obtained in the ninth-order calculation up to eleventh order, we arrive at the expansion of $\tau_M^{(\mathrm{nl})}$ (19).

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